GFD I, Final Exam Solutions

3/7/2012 Parker MacCready

1.(a) The expression for the pressure perturbation is found from the vertical momentum equation:

$$\text{Z-MOM} \qquad w_t = -\frac{1}{\rho_0} p'_z + b$$

which may be rearranged to give:

$$\frac{p'}{\rho_0} = \int \left(b - w_t \right) dz \qquad (*)$$

Note that the first term in the integral gives the hydrostatic part of the pressure, while the second, $-w_t$, may be thought of as the non-hydrostatic "correction" to the basic hydrostatic pressure. We know from class that the wave solution is given by $w = -\delta k U \cos(\varphi)$ where the "phase" φ is given by $\varphi = kx + mz - \omega t$. We also know from the last homework set that $b = -N^2 \delta \sin(\varphi)$. One way to solve (*) is to express w in terms of b using a time-derivative of the DENS equation, which gives: $w_t = \left(-1/N^2\right) b_{tt}$. Hence we may write (*) as

$$\frac{p'}{\rho_0} = \int \left(b + \frac{b_{tt}}{N^2} \right) dz = \frac{N^2 \delta}{m} \left(1 - \frac{\omega^2}{N^2} \right) \cos(\varphi) \quad (**)$$

which is the desired result.

1.(b) Here I am asking you to rewrite the answer above in terms of "external" parameters, like the flow speed and the topographic shape. The dispersion relation for non-rotating waves is given by

$$\omega^2 = U^2 k^2 = \frac{N^2 k^2}{m^2 + k^2}$$

Where we have made use of the intrinsic frequency $\omega = Uk$. This may be solved for m, giving

$$m = -k\left(\frac{1-F^2}{F^2}\right)^{1/2}$$

Where $F \equiv (Uk)/N$, and $F<1$ for wave solutions. We have chosen the negative root to ensure that energy propagation is upwards, as discussed in class. Substituting this expression for m into (**) it is readily verified that

$$\frac{p'}{\rho_0} = -N\delta U\sqrt{1-F^2}\cos(\varphi) \quad (***)$$

1.(c) As the flow speed increases, the wave frequency increases towards N: $Uk \to N$ and so $F \to 1$. For this case (***) shows clearly that the perturbation pressure goes to <u>zero</u>.

1.(d) This is actually a difficult question to answer in a concise way. The simplest might just be to say that (in Z-MOM) the size of the buoyancy term, b, is fixed by the topography and the stratification, whereas the vertical acceleration term increases with U, and so as U increases the two will eventually become equal, and then there is no need for a perturbation pressure. A more complete answer would look at the volume-integrated momentum balance of a fluid parcel, dissecting how exactly the parcel mass times its vertical acceleration is balanced by two forces: (i) the acceleration of gravity times the mass of the parcel, and (ii) the net pressure pushing on the sides of the parcel.

1.(e) The form drag is given by:

$$\begin{aligned}\frac{\text{Force exerted on the fluid by the boundary}}{\text{unit horizontal area}} &= \overline{-p'\big|_{z=0}\frac{\partial z_b}{\partial x}}^{x} \\ &= \rho_0\left(N\delta U\sqrt{1-F^2}\right)(\delta k)\overline{\cos^2\left[k(x-Ut)\right]}^{x} \\ &= \frac{1}{2}\rho_0 N\delta^2 Uk\sqrt{1-F^2}\end{aligned}$$

Positive pressure perturbations are at places with negative topographic slope, meaning that the flow is pushing the topography to the left (negative *x*-direction). Hence the topography must be pushing the flow to the right (positive *x*-direction). This is consistent with the positive sign of the expression above. As I mentioned in class, the issue of *where* in the fluid that change of momentum ends up is more complicated – in steady situations it is where the waves break.

2.(a) In this problem everything is constant in the x-direction, so mass conservation reduces to $v_y = -w_z$. Integrating this vertically through the bottom boundary layer we find:

$$\frac{\partial}{\partial y}\int_0^{\delta^+} v\,dz = \frac{\partial}{\partial y}\left(\text{Ekman transport in the } y\text{-direction}\right) = -\left(w\big|_{z=\delta^+} - \cancel{w\big|_{z=0}}\right) = -w_E$$

2.(b) Using the expression derived in class for the Ekman transport we find

$$w_E = -\frac{A}{f\delta}\frac{\partial U}{\partial y}$$

2.(c) The hydrostatic pressure just above the boundary layer is given by

$$\cancel{p_{ATM}} - p\big|_{z=\delta^+} = -\rho_0 g\left(H + \eta - \delta^+\right)$$

and we may assume that the atmospheric pressure is zero (it could be any constant).

2.(d) The surface height field is given by $\eta = \eta_0 \cos(ly)$ and we may allow the constant η_0 to change slowly with time as the flow "spins-down." We assumed that the fluid velocity (above the boundary layer) was approximately in geostrophic balance with this surface height field, so that

$$U = -\frac{g}{f}\eta_y$$

Thus the Ekman pumping velocity is given by

$$w_E = -\frac{A}{f\delta}\frac{\partial U}{\partial y} = \frac{Ag}{f^2\delta}\eta_{yy}$$

Now, when we evaluate the area integral to find out the pressure work, we are integrating over one wavelength in the y-direction, so any terms that just vary as a cosine will vanish. The term that remains will come from where we have the Ekman pumping velocity (which varies as $\cos(ky)$ times the part of the pressure which also varies as $\cos(ky)$.

This is given by

$$-\int_A (u_n p)\, dA = -\int_A (u_n p)\Big|_{z=\delta^+} dA = \int_0^L \left[\int_0^{2\pi/l} \left(w_E\, p\big|_{z=\delta+} \right) dy \right] dx$$

$$= \left[L\frac{\rho_0 A g^2}{f^2\delta} \right] \int_0^{2\pi/l} \eta_{yy}\eta\, dy = \left[L\frac{\rho_0 A g^2}{f^2\delta} \right] \left\{ \cancel{\int_0^{2\pi/l} \left(\eta_y \eta\right)_y} dy - \int_0^{2\pi/l} \left(\eta_y\right)^2 dy \right\}$$

$$= \left[L\frac{\rho_0 A g^2}{f^2\delta} \right] \left\{ -\frac{1}{2}\left(\frac{2\pi}{l}\right)\eta_0^2 l^2 \right\} = -\frac{\pi\rho_0 L A g^2 \eta_0^2 l}{f^2\delta}$$

2.(e) The above expression is *negative*, meaning that the pressure work is *removing* energy from the overlying volume. This makes sense physically because energy is being dissipated by turbulence in the bottom boundary layer, and the source of energy for this is the KE and APE of the overlying flow. The connection between the overlying flow and the bottom boundary layer is the pressure work.

3.(a) Call the lower layer number 2, and the upper layer number 1. The potential density in the two layers is given by

$$\rho_{1pot} = \frac{p_{ref}}{R\theta_1} = \frac{10^5 \text{ N m}^{-2}}{\left(287 \text{ N m kg}^{-1}\text{ K}^{-1}\right)\left(285 \text{ K}\right)} = 1.2226 \text{ kg m}^{-3}$$

$$\rho_{2pot} = \frac{p_{ref}}{R\theta_2} = \frac{10^5 \text{ N m}^{-2}}{\left(287 \text{ N m kg}^{-1}\text{ K}^{-1}\right)\left(275 \text{ K}\right)} = 1.267 \text{ kg m}^{-3}$$

Note that the potential temperature *increases* from the lower layer to the upper layer. Thus the potential density jump is given by

$$\Delta\rho_{pot} = \rho_{2pot} - \rho_{1pot} = 0.0445 \text{ kg m}^{-3}$$

And the reduced gravity is

$$g' = \frac{g\Delta\rho_{pot}}{\rho_{2pot}} = \frac{\left(9.8 \text{ m s}^{-2}\right)\left(0.0445 \text{ kg m}^{-3}\right)}{1.267 \text{ kg m}^{-3}} = 0.344 \text{ m s}^{-2}$$

3.(b) The effective depth is given by

$$H_{eff} = \frac{H_1 H_2}{H_1 + H_2} \underset{\lim H_1\to\infty}{=} H_2 = 800 \text{ m}$$

3.(c) The Kelvin wave speed will be to the north. For the 1-layer waves we covered in class, that wave speed is $\sqrt{gH}$, but because this is a 2-layer problem the speed will be given by

$$c = \sqrt{g' H_{eff}} = \left[\left(0.344 \text{ m s}^{-2}\right)\left(800 \text{ m}\right) \right]^{1/2} = 16.6 \text{ m s}^{-1}$$

3.(d) The internal Rossby Radius of deformation (covered in Problem Set 4) is given by

$$a' = \frac{c}{f\left(45° \text{ latitude}\right)} = \frac{16.6 \text{ m s}^{-1}}{2 \times 7.292 \times 10^{-5} \sin\left(45°\right) \text{ s}^{-1}} = 161 \text{ km}$$

3.(e) The maximum northward wind speed can be calculated from the 2-layer Y-MOM equation (with $u = 0$ because it is at the coastal boundary and $\eta = 0$ because the upper layer is infinitely thick):

$$\frac{\partial v_2}{\partial t} = -g' \frac{\partial E}{\partial y}$$

The interface shape at the coast will be of the form $E = E_0 \cos\left[l\left(y - ct\right) \right]$ with $E_0 = 400 \text{ m}$. Thus it is easy to show that

$$v_2 = \frac{g' E_0}{c} \cos\left[l\left(y - ct\right) \right] = c \frac{E_0}{H_2} \cos\left[l\left(y - ct\right) \right]$$

And this has a maximum northward windspeed (under peaks of the interface at the coast) of

$$v_{2\max} = c \frac{E_0}{H_2} = \frac{c}{2} = 8.3 \text{ m s}^{-1}$$